RAYLEIGH STABILITY OF SHEAR FLOW IN RELAXING MEDIA

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The stability of steady-state flow is considered in a medium with a nonlocal coupling between pressure and density. The equations for perturbations in such a medium are derived in the linear approximation. The results of numerical integration are given for shear motion. The stability of parallel layered flow in an inviscid homogeneous fluid has been studied for a hundred years. The mathematics for investigating an inviscid instability has been developed, and it has been given a physical interpretation. The first important results in flow stability of an incompressible fluid were obtained in the papers of Helmholtz, Rayleigh, and Kelvin [1] in the last century. Heisenberg [2] worked on this problem in the 1920's, and a series of interesting papers by Tollmien [3] appeared subsequently. Apparently one of the first problems in the stability of a compressible fluid was solved by Landau [4]. The first investigations on the boundary-layer stability of an ideal gas were carried out by Lees and Lin [5], and Dunn and Lin [6]. Mention should be made of a series of papers which have appeared quite recently [7-9]. In all the papers mentioned flow stability is investigated in the framework of classical single-phase hydrodynamics. Meanwhile, in recent years, the processes by which perturbations propagate in media with relaxation have been intensively studied [10-12].

1. Fundamental Equations

We consider the problem of the stability of relatively small perturbations in the steady-state flow of a fluid with the following equation of state:

$$\delta p = c_0^2 \delta\rho + \beta d\delta\rho/dt + \varkappa d^2\delta\rho/dt^2, \tag{1.1}$$

where δp and $\delta\rho$ are small perturbations of pressure and density. We shall neglect the effect of viscosity not caused by internal processes.

Let the unperturbed flow be defined by the relations

$$p_0=\text{const},\quad \text{div } V=0,\quad V_x=V(y).$$

We shall assume that the flow parameters oscillate about these quantities

$$\begin{gathered} v_x=V(y)+\tilde{u}(x,\ y,\ z,\ t);\\ v_y=\tilde{v}(x,\ y,\ z,\ t);\\ v_z=\tilde{W}(x,\ y,\ z,\ t);\\ p=p_0+\tilde{p}(x,\ y,\ z,\ t);\\ \rho=\rho_0+\tilde{\rho}(x,\ y,\ z,\ t), \end{gathered} \tag{1.2}$$

where the sign ~ refers to the small fluctuating quantities. Writing the Euler equation for Eqs. (1.2) and neglecting quantities of the second order of smallness in the perturbations and their derivatives we obtain the following system of partial differential equations:

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$$\partial\tilde{u}/\partial t + V\partial\tilde{u}/\partial x + \tilde{v}V' = -(1/\rho_0)\partial\tilde{p}/\partial x;$$
$$\partial\tilde{v}/\partial t + V\partial\tilde{v}/\partial x = -(1/\rho_0)\partial\tilde{p}/\partial y;$$
$$\partial\tilde{W}/\partial t + V\partial\tilde{W}/\partial x = -(1/\rho_0)\partial\tilde{p}/\partial z; \tag{1.3}$$
$$\partial\tilde{\rho}/\partial t + V\partial\tilde{\rho}/\partial x + \rho_0(\partial\tilde{u}/\partial x + \partial\tilde{v}/\partial y + \partial\tilde{W}/\partial z = 0,$$

where the prime denotes differentiation with respect to y. The equation of state (1.1) is rewritten in the form

$$\tilde{p} = c_0^2\tilde{\rho} + \beta(\partial\tilde{\rho}/\partial t + V\partial\tilde{\rho}/\partial x) + \varkappa(\partial^2\tilde{\rho}/\partial t^2 + 2V\partial^2\tilde{\rho}/\partial t\partial x + V^2\partial^2\tilde{\rho}/\partial x^2). \tag{1.4}$$

The system (1.3), (1.4) has solutions which are exponential functions of the variables x, z, t. We shall look for these solutions in the form

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{W} \\ \tilde{\rho} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} u(y) \\ v(y) \\ W(y) \\ \rho(y) \\ p(y) \end{pmatrix} \exp[i\alpha(x - ct) + i\gamma z]. \tag{1.5}$$

In what follows we shall restrict ourselves to investigating perturbations which are periodic in the spatial coordinates x and z. Consequently, in this case α and γ in Eq. (1.5) must be real, and c can be complex,

$$c = c_r + ic_i. \tag{1.6}$$

If αc_i in Eq. (1.6) is greater than zero, then the perturbations Eqs. (1.5) increase with time. If αc_i is less than zero, the perturbations are damped.

Substituting Eq. (1.5) in Eqs. (1.3), (1.4), we obtain a system of ordinary differential equations in y,

$$i\alpha(V - c)u + vV' = -(i\alpha/\rho_0)p; \quad i\alpha(V - c)v = -(1/\rho_0)p';$$
$$i\alpha(V - c)W = -(i\gamma/\rho_0)p;$$
$$(i\alpha/\rho_0)(V - c)\rho + i\alpha u + v' + i\gamma W = 0; \tag{1.7}$$
$$p = c_0^2\rho + i\alpha\beta(V - c)\rho - \varkappa\alpha^2(V - c)^2\rho.$$

Let V_{max} by the characteristic velocity of the unperturbed flow, and h its characteristic dimension. We introduce the dimensionless quantities

$$\bar{\rho} = \rho/\rho_0, \quad \bar{p} = p/\rho_0 V_{max}^2, \quad \bar{V} = V/V_{max}, \quad \bar{u} = u/V_{max},$$
$$\bar{c}_0 = c_0/V_{max} = 1/M, \quad \bar{x} = x/h, \quad \bar{y} = y/h, \quad \bar{z} = z/h,$$
$$\bar{\gamma} = \gamma h, \quad \bar{\alpha} = \alpha h, \quad \bar{v} = v/V_{max}, \quad \bar{W} = W/V_{max}, \tag{1.8}$$
$$\bar{\beta} = \beta/hV_{max}, \quad \bar{\varkappa} = \varkappa/h^2,$$

where M is the Mach number.

Using Eq. (1.8) we rewrite Eq. (1.7) omitting the bar above the dimensionless quantities:

$$i\alpha(V - c)u + vV' = -i\alpha p,$$
$$i\alpha(V - c)v = -p',$$
$$i\alpha(V - c)W = -i\gamma p, \tag{1.9}$$
$$i\alpha(V - c)\rho + i\alpha u + v' + i\gamma W = 0,$$
$$p = (1/M^2)[1 + i\alpha\beta M^2(V - c) - \varkappa\alpha^2 M^2(V - c)^2]\rho.$$

Thus, the problem of the stability of steady-state flow reduces to finding the eigenvalues of c for the system (1.9) which satisfy the specific boundary conditions of the perturbation. It can be shown that the stability problem of three-dimensional perturbations is equivalent to the stability problem of two-dimensional perturbations with a smaller Mach number and a larger parameter β. Thus, we can restrict the treatment to two-dimensional perturbations.

2. Derivation of the Equations for Two-Dimensional Velocity and Pressure Perturbations

We now set W=0 and $\gamma=0$ in the system (1.9). The quantities p and ρ are eliminated from the first two equations of system (1.9) by using the last two equations,

$$i\alpha(V-c)u+vV'=(i\alpha u+v')\frac{1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2}{M^2(V-c)}; \tag{2.1}$$

$$i\alpha(V-c)v=\left[\frac{1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2}{i\alpha M^2(V-c)}(i\alpha u+v')\right]'. \tag{2.2}$$

We now use Eq. (2.1) to express u in terms of v:

$$i\alpha u=\left[\frac{1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2}{M^2(V-c)}v'-vV'\right]\frac{M^2(V-c)}{(V-c)^2M^2(1+\varkappa\alpha^2)-i\alpha\beta M^2(V-c)-1}. \tag{2.3}$$

Substituting Eq. (2.3) in Eq. (2.2), we obtain

$$v''-v'[V'(V-c)M^2(2+i\alpha\beta M^2(V-c))/\Delta]+v[-\alpha^2-V''/(V-c)+$$
$$+\alpha^2M^2(V-c)^2/(1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2)+V'^2M^2(2+i\alpha\beta M^2(V-c))/\Delta]=0,$$
$$\Delta=[1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2]\cdot[M^2(V-c)^2(1+\varkappa\alpha^2)-i\alpha\beta M^2(V-c)-1]. \tag{2.4}$$

If M=0, then Eq. (2.4) passes to Rayleigh's equation, defining the stability of parallel flows of an inviscid incompressible fluid.

If $\beta=\varkappa=0$, $M\neq0$, then Eq. (2.4) coincides with formula (5.3.20) in [13] if we set T=1 in the latter.

We return once again to the system (1.9). We express v, u, and ρ in terms of p:

$$\rho=\{M^2/[1+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2]\}\,p;$$
$$v=[i/\alpha(V-c)]p';$$
$$u=-[1/(V-c)]p-[V'/\alpha^2(V-c)^2]p'. \tag{2.5}$$

Substituting Eq. (2.5) in the fourth equation of system(1.9), we obtain the equation for p:

$$p''-[2V'/(V-c)]p'-\alpha^2p\{1-M^2(V-c)^2/[1+$$
$$+i\alpha\beta M^2(V-c)-\varkappa\alpha^2M^2(V-c)^2]\}=0. \tag{2.6}$$

If $\beta=\varkappa=0$, then Eq. (2.6) coincides with the equation for p in the case of a compressible fluid (see, for example, [8]). For flows which are bounded by hard walls, Eqs. (2.4)-(2.6) are solved with the boundary conditions

$$v|_{y=l}=0;\ dp/dy|_{y=l}=0,$$

where l is the y coordinate of the wall.

For free flows, allowing for the fact that V'=0 at infinity, the following boundary conditions can be obtained from Eqs. (2.4), (2.6) for perturbations which increase with time:

$$v|_{y=\pm\infty}=0,\ p|_{y=\pm\infty}=0. \tag{2.7}$$

If $\beta=0$ and

$$1/\alpha M\varkappa^{1/2}>|V(\infty)-c|>1/M(1+\varkappa\alpha^2)^{1/2},$$

then perturbations which are neutral in time do not vanish at infinity and Eq. (2.7) is not satisfied. In this case the boundary conditions reduce to the requirement that the perturbations at infinity should depart.

3. Investigations of Shear Flow Stability

The stability problem was solved numerically for the basic flow profile

$$V(y)=\text{th}\,y. \tag{3.1}$$

Eigenvalues c were sought for Mach numbers less than unity. In order not to encounter the difficulties arising when $c_i=0$, Eqs. (2.4) and (2.6) were solved only with those values of α for which c_i was still nonzero for the given Mach number.

TABLE 1

M	0	0,2	0,4	0,8
α	c_i			
0,05	0,914664	0,880269	0,792673	0,547393
0,1	0,836440	0,805063	0,723345	0,480605
0,15	0,764315	0,735587	0,659242	0,419158
0,2	0,697455	0,671047	0,599557	0,362019
0,3	0.576895	0,554264	0.491015	0,257844
0,4	0.470451	0.450676	0.393933	0,164127
0,5	0,375022	0,357401	0.305719	0,078692
0,52				0,062465
0,54				0,046496
0,56				0,030772
0,6	0.288324	0.272315	0.224518	
0,7	0,208643	0,193847	0.148956	
0,75			0.112951	
0,8	0,134669	0.120758	0.077983	
0.85	0,099496	0,085928	0.043955	
0,9	0,065387	0,052105		
0,95	0.032250	0,019202		

TABLE 2

	c_i			
	M=0,4		M=0,8	
α	ϰ=0,1	ϰ=0,2	ϰ=0,1	ϰ=0,2
0,1	0.723341	0.723337	0.480556	0.480508
0,2	0.599535	0.599513	0,361682	0,361346
0,3	0.490963	0.490911	0.256814	0.255778
0.4	0.393843	0.393754	0.161882	0.159578
0.5	0.305589	0.305460	0.074781	0.070655
0.52	0,		0.058198	0.053676
0.54			0.041877	0.036960
0.56			0.025809	0.020508
0.6	0.224349	0.224179	0,	
0.7	0.148748	0.148538		
0.75	0.112724	0.112495		
0.8	0.077737	0.077488		
0,85	0,043691	0.043423		

TABLE 3

	c_i			
	M=0,2		M=0,8	
α	β=1	β=10	β=1	β=10
0,1	0.805154	0.805990	0.496011	0,610520
0,2	0,671166	0,672289	0.388198	0.542480
0,3	0.554383	0,555553	0.291754	0,460170
0.4	0,450782	0,451899	0,203215	0,377236
0.5	0,357489	0.358516	0,120289	0.297443
0.6	0.272393	0.273320	0.041324	0,221685
0.7	0,193899	0,194734		
0.8	0.120793	0,121544		

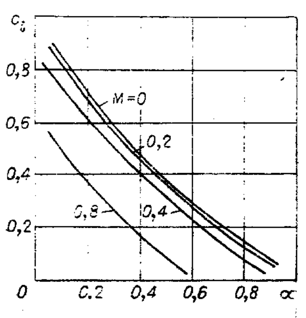

Fig. 1

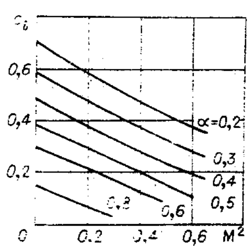

Fig. 2

On making the substitution

$$\varphi = g'/g \tag{3.2}$$

the homogeneous linear second-order equations (2.4) and (2.6) reduce to the nonlinear Riccati equation

$$\varphi' + \varphi^2 + a\varphi + b = 0. \tag{3.3}$$

Here g is the velocity if we take Eq. (2.4), or the pressure if we take Eq. (2.6); a and b are the coefficients of the first and zeroth derivatives in the corresponding equations. The boundary conditions for φ have the form

$$\varphi|_{y=-\infty} = \alpha\{1 - M^2(1+c)^2/[1 - i\alpha\beta M^2(1+c) - \varkappa\alpha^2 M^2(1+c)^2]\}^{1/2}; \tag{3.4}$$

$$\varphi|_{y=+\infty} = -\alpha\{1 - M^2(1-c)^2/[1 + i\alpha\beta M^2(1-c) - \varkappa\alpha^2 M^2(1-c)^2]\}^{1/2}. \tag{3.5}$$

Here the real part of the root is larger than zero.

Equation (3.3) was solved numerically by the Runge–Kutta method in two ways. In the first the boundary conditions (3.4), (3.5) were transferred from infinity to the finite distances $y_1 = -l$ and $y_2 = l$.

For c equal to some c_n the integration of Eq. (3.3) began from y_1 with the initial condition (3.4), and the value of φ obtained at the point y_2 was then compared with that from Eq. (3.5). A new value of c was calculated using the method of secants

$$c_{n+1} = c_n + [(c_n - c_{n-1})(\varphi_2(c_n) - \varphi_n(c_n))]/[\varphi_n(c_n) - \varphi_n(c_{n-1})], \tag{3.6}$$

where $\varphi_2(c_n)$ is the boundary condition (3.5) calculated for $c = c_n$; $\varphi_n(c_n)$ and $\varphi_n(c_{n-1})$ are the values of φ at the point y_2 obtained from the solution of Eq. (3.3) for $c = c_n$ and $c = c_{n-1}$.

The process was continued until

$$\mathrm{mod}\ (c_n - c_{n-1}) > \varepsilon. \tag{3.7}$$

It can be seen from Eq. (3.6) that we must have two approximate values of c to solve the problem. These were taken from results previously obtained. To check the correctness of the calculations the quantity c was found independently from Eqs. (2.4), (2.6) by the same method for various values of l. When solving, ε was taken equal to 10^{-6} in Eq. (3.7), and the integration step was taken to be $\Delta y = 0.01$.

Numerical integration showed that $c_r = 0$ always. It was found that for $l = 3$

$$\max |c_1 - c_2| \sim 10^{-2},$$

for $l = 6$

$$\max |c_1 - c_2| \sim 10^{-5},$$

where c_1 is the imaginary part of c, found by solving Eq. (2.4), and c_2 is the imaginary part of c found by solving Eq. (2.6).

Only Eq. (2.6) was solved by the second method. The following substitution is made for the independent variable: $z = \tanh y$ and from Eqs. (2.6), (3.2), and (3.3) we obtain for the profile (3.1)

$$(1 - z^2)\varphi' + \varphi^2 - [2(1-z^2)/(z-c)]\varphi - \alpha^2[1 - M^2(z-c)^2/(1 + i\alpha\beta M^2(z-c) - \varkappa\alpha^2 M^2(z-c)^2)] = 0, \tag{3.8}$$

where the prime denotes differentiation with respect to z. The boundary conditions for φ (3.4), (3.5) are taken at the points $z = 1$, $z = -1$.

It is clear from Eqs. (3.4), (3.5), and (3.8) that the value of φ' at these points is indeterminate. Developing this indeterminacy according to l'Hôpital's rule, we have

$$\varphi'(z = -1) = \left\{(1+\varphi)^{-1}\left[-\frac{2\varphi}{1+c} + \frac{\alpha^2 M^2(1+c)(1 - 0{,}5i\alpha\beta M^2(1+c))}{[1 - i\alpha\beta M^2(1+c) - \varkappa\alpha^2 M^2(1+c)^2]^2}\right]\right\}_{z=-1};$$

$$\varphi'(z = 1) = \left\{(1-\varphi)^{-1}\left[\frac{2\varphi}{1-c} + \frac{\alpha^2 M^2(1-c)(1 + 0{,}5i\alpha\beta M^2(1-c))}{[1 + i\alpha\beta M^2(1-c) - \varkappa\alpha^2 M^2(1-c)^2]^2}\right]\right\}_{z=1}.$$

The process of searching for c is similar to that described in the first method. The difference between the values of c obtained from Eq. (2.6) as solved by the first method with $l = 6$, and by the second method was less than 10^{-6}, and so we can have confidence in c to five significant figures.

For M = 0 and allowing for the discrepancy in the normalization of V, the values of c_i coincide with the results of [14], where the c_i are given with an accuracy to four significant figures.

The values of c_i found for $\beta=\varkappa=0$, $M\neq 0$ are given in Fig. 1 and in Table 1 as functions of the wave number α.

Figure 2 shows how the quantity c_i changes as a function of M^2 for a fixed value of α. Within the limits of accuracy of the graph this function is linear for small M^2, and so by using the values of c_i obtained previously we were able to set the initial approximations for c_i successfully for the other Mach numbers. It is clear from this that for shear flow the compressibility turns out to have a stabilizing action on growing perturbations. This agrees with already familiar results (see, for example, [8]).

Values of c_i are given in Table 2 for fixed values of M, $\beta=0$, and $\varkappa\neq 0$. It is clear that the flow becomes more stable as $\varkappa$ increases.

Values of c_i are given in Table 3 for $\beta\neq 0$, $\varkappa=0$. In this case the instability is amplified.

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